

Notes

Calendar

| 2025-10-01 – 2025-10-31 | | | | | | |
|-------------------------|-----|-----|-----|-----|-----|-----|
| Mon | Tue | Wed | Thu | Fri | Sat | Sun |
| | | 1 | 2 | 3 | 4 | 5 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| 20 | 21 | 22 | 23 | 24 | 25 | 26 |
| 27 | 28 | 29 | 30 | 31 | | |

| 2025-11-01 – 2025-11-30 | | | | | | |
|-------------------------|-----|-----|-----|-----|-----|-----|
| Mon | Tue | Wed | Thu | Fri | Sat | Sun |
| | | | | | 1 | 2 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| 17 | 18 | 19 | 20 | 21 | 22 | 23 |
| 24 | 25 | 26 | 27 | 28 | 29 | 30 |

| 2025-12-01 – 2025-12-31 | | | | | | |
|-------------------------|-----|-----|-----|-----|-----|-----|
| Mon | Tue | Wed | Thu | Fri | Sat | Sun |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 8 | 9 | 10 | 11 | 12 | 13 | 14 |
| 15 | 16 | 17 | 18 | 19 | 20 | 21 |
| 22 | 23 | 24 | 25 | 26 | 27 | 28 |
| 29 | 30 | 31 | | | | |

| 2026-01-01 – 2026-01-31 | | | | | | |
|-------------------------|-----|-----|-----|-----|-----|-----|
| Mon | Tue | Wed | Thu | Fri | Sat | Sun |
| | | | 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| 19 | 20 | 21 | 22 | 23 | 24 | 25 |
| 26 | 27 | 28 | 29 | 30 | 31 | |

| 2026-02-01 – 2026-02-28 | | | | | | |
|-------------------------|-----|-----|-----|-----|-----|-----|
| Mon | Tue | Wed | Thu | Fri | Sat | Sun |
| | | | | | | 1 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 16 | 17 | 18 | 19 | 20 | 21 | 22 |
| 23 | 24 | 25 | 26 | 27 | 28 | |

| 2026-03-01 – 2026-03-31 | | | | | | |
|-------------------------|-----|-----|-----|-----|-----|-----|
| Mon | Tue | Wed | Thu | Fri | Sat | Sun |
| | | | | | | 1 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 16 | 17 | 18 | 19 | 20 | 21 | 22 |
| 23 | 24 | 25 | 26 | 27 | 28 | 29 |
| 30 | 31 | | | | | |

| | |
|------------------|-----------------------------|
| Days total: | 168 |
| Days past: | 131 |
| Days left: | 37 |
| Time left: | duration(weeks: 5, days: 2) |
| Percentage left: | 0.22023809523809523 |

Abstract

• Friedl Lück

• Yi Liu:

1. Introduction

▸ L^2 -Alexander Torsion

– $\tau^{(2)} : \text{Admissible Triple} \rightarrow \frac{\mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}}{\cong}$

$$\tau^{(2)}(N, \varphi, \gamma)(t) = ???$$

• $f \stackrel{\cong}{\sim} g \Leftrightarrow \exists r \in \mathbb{R} \forall t \in \mathbb{R}_{>0} : f(t) = t^r g(t)$

• “getwistete” variante der full L^2 -Alexander Torsion

– $N : 3$ -Mannifold

• needed: (empty-/incompressible toral-) boundary

• in [1]: prime

– Admissible Triple := (

$N,$

$$\varphi : H^1(N, \partial N; \mathbb{R}) \stackrel{[1]}{\underset{\text{Grp}}{=} \pi_1(N)} \rightarrow \mathbb{R},$$

$$\gamma : \pi_1(N) \underset{\text{Grp}}{\rightarrow} G,$$

$$h_1 : \text{Prop} = (\exists \tilde{\varphi} : \pi_1(N) \underset{\text{Grp}}{\xrightarrow{\gamma}} G \underset{\text{Grp}}{\xrightarrow{\tilde{\varphi}}} \mathbb{R} = \pi_1(N) \underset{\text{Grp}}{\xrightarrow{\varphi}} \mathbb{R})$$

▸ full L^2 -Alexander Torsion with respect to φ

– $\tau^{(2)}(N, \varphi)(t)$

read through
Friedl Lück

- if $\gamma = \text{id}_{\pi_1(N)}$
- ▶ **Zu zeigen:**
 - $\tau^{(2)}(N, \varphi)(t)$
 - continuous
 - positive
 - asymptotically monomial in both ends
 - ▶ $f : \mathbb{R}_{>0} \rightarrow [0, +\infty)$

$$\exists C_{+\infty}, d_{+\infty} \in \mathbb{N} : \lim_{t \rightarrow +\infty} \frac{f(t)}{t^{d_{+\infty}}} = C_{+\infty} \text{ monomial in the limit } t \rightarrow +\infty$$

$$\exists C_{0+}, d_{0+} \in \mathbb{N} : \lim_{t \rightarrow 0+} \frac{f(t)}{t^{d_{0+}}} = C_{0+} \text{ monomial in the limit } t \rightarrow 0+ \quad [1]$$
 - $\text{deg} \tau^{(2)}(N, \varphi)(t)$ is Thurston norm of φ
 - $\text{deg}^a(f) = d_{+\infty} - d_{0+}$
 - ▶ examples:
 - $\frac{t^2}{t} \xrightarrow{\text{deg}^a} 2 - 1 = 0$
 - ▶ $\frac{e^{-t}}{t^p} \xrightarrow{\text{deg}^a} 0 - \underbrace{(-p)} = p$

$$\lim_{t \rightarrow 0+} \frac{\sum_{n=0}^{\infty} \frac{(-t)^n}{n!}}{t^{d_{0+}}}$$
- ▶ $\tau^{(2)}(N, \varphi, \gamma : \pi_1(N) \rightarrow G)(t)$
 - continuous
 - positive (if G *residually finite* & (N, γ) *weakly acyclic*)
 - $\widetilde{\text{deg}} \tau^{(2)}(N, \varphi, \gamma : \pi_1(N) \rightarrow G)(t)$ bounded by Thurston norm of φ
 - generalized degree
 - $\text{deg}_{+\infty}^b(f) = \inf \left\{ D_{+\infty} \in \mathbb{R} : \lim_{t \rightarrow +\infty} \frac{f(t)}{t^{D_{+\infty}}} = 0 \right\}$
 - $\text{deg}_{0+}^b(f) = \sup \left\{ D_{0+} \in \mathbb{R} : \lim_{t \rightarrow 0+} \frac{f(t)}{t^{D_{0+}}} = 0 \right\}$
 - $\text{deg}^b(f) = \text{deg}_{+\infty}^b(f) - \text{deg}_{0+}^b(f)$ *growth bound degree*¹

Definition of deg?

Chapters (Liu)

1. Introduction
2. Preliminaries
3. Regular Fugledge-Kadison determinant
4. Multiplicatively convex functions
5. Multiplicative convexity and exponent bound
6. Continuity of degree
7. asymptotics for integral matrices
8. L^2 -Alexander torsion of 3-manifolds
9. examples

Chapters (Masters thesis)

1. torsion invariants
 1. Alexander polynomial (Knots)
 2. Reidemeister torsion $\rho(X)$ / Alexander torsion $\tau(K)$
 3. L^2 -Torsion invariants
 1. twisting (see: *full* \rightsquigarrow L^2 -Alexander Torsion)
2. thurston norm
3. connection
 1. asymptotic degree

Unterschied?

¹Definition 1.3 aus Yi Liu

Reidemeister torsion

Definition the **determinant** is defined as::

- A a free R -module of rank n
- $T : A \xrightarrow{R\text{-mod}} A$
 - ▶ $\sim \triangleright$

$$\bigwedge^n T : \bigwedge^n A \xrightarrow{R\text{-mod}} \bigwedge^n A$$

$\det : \{n : \mathbb{N}\} \rightarrow \{A : R\text{-mod}\} \rightarrow \{\text{rank } A = n\} \rightarrow$

$$\left(A \xrightarrow{R\text{-mod}} A \right)_{R\text{-mod}} \rightarrow R$$

$\det(T) = r$

where $\forall v_1, \dots, v_n \in A : \left(\bigwedge^n T \right) (v_1 \wedge \dots \wedge v_n) = r \cdot (v_1 \wedge \dots \wedge v_n)$

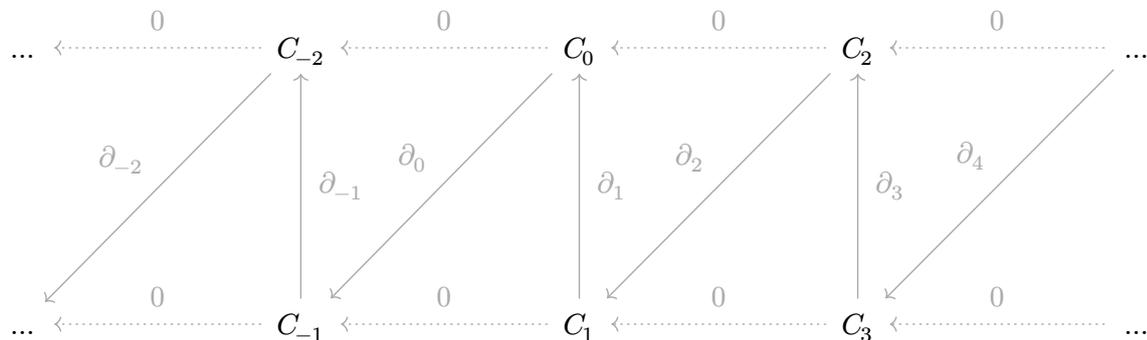
Definition Chain complexes are defined as::

$(C_*, \partial_*) : \text{Chain}_{R\text{-mod}}$

$$\dots \xleftarrow{\partial_{-2}} C_{-2} \xleftarrow{\partial_{-1}} C_{-1} \xleftarrow{\partial_0} C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \dots$$

$$\forall i \in \mathbb{Z} : \partial_{i+1} \circ \partial_i = 0$$

The requirement $\partial_{i+1} \circ \partial_i = 0$ makes writing the chain complex in the following way “obvious”:



Definition We define the **Reidemeister-Torsion** as::

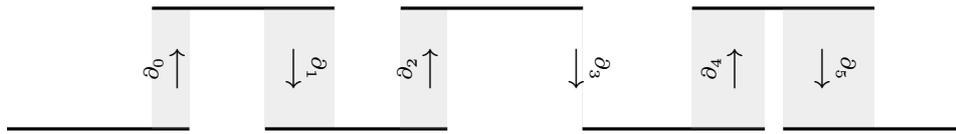
- $C_* : \text{Chain}_{R\text{-mod}}$
 - ▶ $\forall i : C_i$ is a free R -module of finite rank n_i
 - $C_i = 0$ for almost all $i \in \mathbb{Z}$
 - C_* contractible
- $\rho : \{\forall i : \text{rank } C_i < \infty\} \rightarrow (C_* : \text{Chain}_{R\text{-mod}}) \rightarrow R$
- ρ

Beer coaster trick

For intuition we will think of each chain module as a beer coaster. We will lay them next to each other as in the picture below:

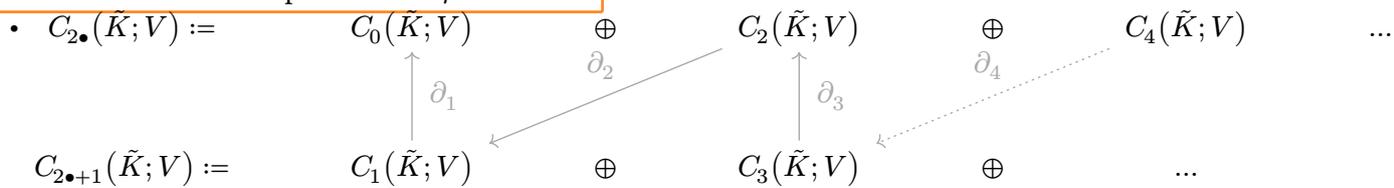


The even ones are on the bottom, and the odd ones on the top. How much the coasters overlap corresponds to how much the differentials transport from one module onto the next.



Definition

- $G \in \text{Grp}$
- finite $\mathbb{Z}G$ chain complex $C_0(\tilde{K}; V) \xleftarrow{\partial_1} \dots \xleftarrow{\partial_m} C_m(\tilde{K}; V)$
 - $C_*(\tilde{K}; V) = V \otimes_{\mathbb{Z}G} C_*(\tilde{K})$
 - $V \cong \mathbb{R}^n$ with representation $\rho : \mathbb{Z}G \curvearrowright V$



there is an isomorphism between $C_{2\bullet+1}$ and $C_{2\bullet}$, if the chain is acyclic, meaning each homology group is 0, or equivalently the coasters overlap completely

Definition²

- K compact CW-complex
 - $\chi(K) = 0$
 - $n \in \mathbb{N}$
 - $\rho : \pi_1 K \rightarrow \text{SL}_n(\mathbb{C})$ representation
- \tilde{K} universal cover
 - $\pi_1 K \curvearrowright \tilde{K}$ deck transform
 - $C_*(\tilde{K}; \mathbb{Z})$ singular chain complex

twisted chain complex:

$$C_*(\tilde{K}; \rho) := C_*(\tilde{K}; \mathbb{Z}) \otimes_{\rho} \mathbb{C}^n$$

$$= \frac{C_*(\tilde{K}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^n}{\underbrace{v \otimes \lambda^{-1} w - \rho(\lambda)v \otimes w}_{C_*(\tilde{K}; \mathbb{Z}) \otimes_{\pi_1 K} \mathbb{C}^n}}$$

$$= \frac{C_*(\tilde{K}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^n}{\underbrace{\lambda^{-1}v \otimes w - v \otimes \rho(\lambda)w}}$$

But where does λ^{-1} live in? Or: what is $C_*(\tilde{K}; \mathbb{Z})$? Apparently $\lambda^{-1} \in \pi_1(K)$, but why?

²<https://de.wikipedia.org/wiki/Reidemeister-Torsion#Konstruktion>

ok to assume start at 0?

please provide some examples

make this align, each line must be of varying sizes

beliebig?

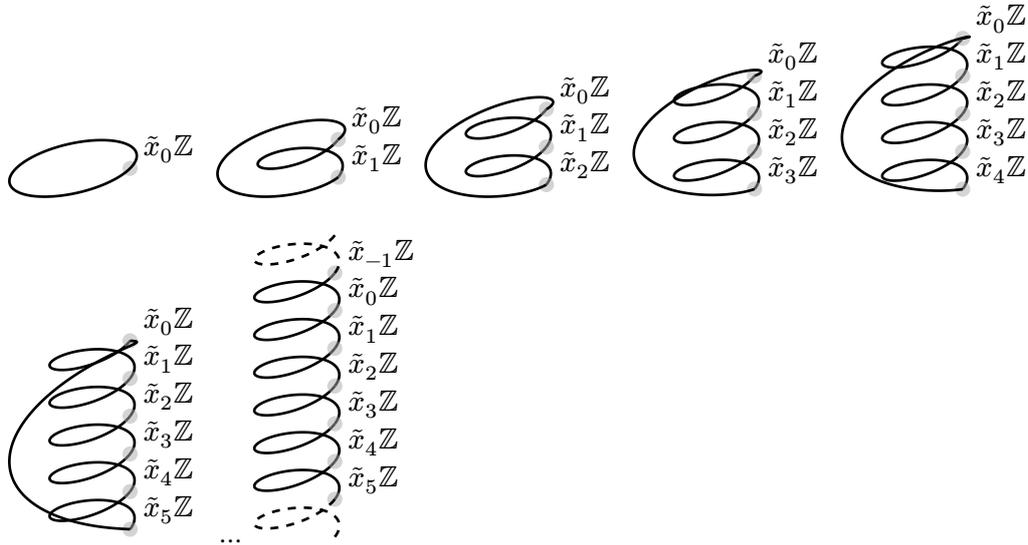
ergibt doch keinen Sinn, dass $\lambda^{-1} w, \pi_1 K \mathbb{C}^n$ weil das ja nicht in \mathbb{C}^n ist?

ich vermute, dass es so sein sollte?

$$C_*(K; \mathbb{Z}) = \underbrace{C_0(K; \mathbb{Z})}_{\bullet} \xleftarrow{\partial_1} \underbrace{C_1(K; \mathbb{Z})}_{\triangle} \xleftarrow{\partial_2} \underbrace{C_2(K; \mathbb{Z})}_{\triangle} \xleftarrow{\partial_3} \dots$$

$$\mathbb{Z}(0 \ 0 \ 0) \oplus \mathbb{Z}(1 \ 0 \ 0) \oplus \mathbb{Z}(0 \ 1 \ 0) \quad \mathbb{Z} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \oplus \mathbb{Z} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \oplus \mathbb{Z} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \mathbb{Z} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For \tilde{K} , we would have $\#C_i(\tilde{K}; \mathbb{Z}) = \infty$ for most i .



$$\text{This makes } \underbrace{C_0(\tilde{K}; \mathbb{Z})}_{\cong \mathbb{Z}^{\pi_1 K}} \xleftarrow{\partial_1} \underbrace{C_1(\tilde{K}; \mathbb{Z})}_{\cong \mathbb{Z}^{\pi_1 K}} \xleftarrow{\partial_2} 0$$

³https://en.wikipedia.org/wiki/Exponential_object

Twisted tensor products

We have:

wie besser
aufschreiben?

$$C_*^{(2)}(\tilde{X}; \kappa) : \text{Chain}(\text{Hilbert-Mod}_{\text{FinGen}})$$

$\otimes_{\kappa(\varphi, t)}$?

$$C_*^{(2)}(\tilde{X}; \kappa) := \ell^2\pi \otimes_{\kappa(\varphi, t)} C_*(\tilde{X})$$

We need

$$\begin{aligned} \otimes & : \text{Hilbert-Mod}_{\text{FinGen}} \\ & \rightarrow \left(\left\{ \pi \xrightarrow{\text{Grp}} \mathbb{R} \right\} \rightarrow \mathbb{R}_{>0} \rightarrow \left\{ \mathbb{Z}\pi \xrightarrow{\text{Ring}} \mathbb{R}\pi \right\} \right) \\ & \rightarrow \text{Hilbert-Mod}_{\text{FinGen}} \\ & \rightarrow \text{Hilbert-Mod}_{\text{FinGen}} \end{aligned}$$

for $\kappa(\varphi, \gamma, t)(g)$ or

$$\begin{aligned} \otimes & : \text{Hilbert-Mod}_{\text{FinGen}} \\ & \rightarrow \left(\left\{ \pi \xrightarrow{\text{Grp}} \mathbb{R} \right\} \rightarrow \left\{ \pi \xrightarrow{\text{Grp}} G \right\} \rightarrow \mathbb{R}_{>0} \rightarrow \left\{ \mathbb{Z}\pi \xrightarrow{\text{Ring}} \mathbb{R}\pi \right\} \right) \\ & \rightarrow \text{Hilbert-Mod}_{\text{FinGen}} \\ & \rightarrow \text{Hilbert-Mod}_{\text{FinGen}} \end{aligned}$$

for $\kappa(\varphi, t)(g)$. Because from context we are provided with φ , γ and t we need

$$\begin{aligned} \otimes & : \text{Hilbert-Mod}_{\text{FinGen}} \\ & \rightarrow \left\{ \mathbb{Z}\pi \xrightarrow{\text{Ring}} \mathbb{R}\pi \right\} \\ & \rightarrow \text{Hilbert-Mod}_{\text{FinGen}} \\ & \rightarrow \text{Hilbert-Mod}_{\text{FinGen}} \end{aligned}$$

and from this, we have⁴:

$$\begin{aligned} \otimes_{(\cdot)} & : (\text{mod-}R) \rightarrow \left(L \xrightarrow{\text{Ring}} R \right) \rightarrow (L\text{-mod}) \xrightarrow{\text{Ab}} \text{Ab} \\ A \otimes_{\kappa} B & = \frac{A \otimes B}{v \otimes \lambda^{-1}w = \kappa(\lambda)v \otimes w} \end{aligned}$$

the “twisted tensor” product.

fix this in
my
definition!

actually $\rho : \pi_1 K \rightarrow \text{SL}_n(\mathbb{C})$ in the article⁵

⁴<https://de.wikipedia.org/wiki/Reidemeister-Torsion#Konstruktion>

⁵<https://de.wikipedia.org/wiki/Reidemeister-Torsion#Konstruktion>

Mainly used in *Reidemeister-Torsion* for chain-complexes:

$$C_*(N, \kappa) := C_*(\tilde{N}, \mathbb{Z}) \otimes_{\kappa} \mathbb{C}^n$$

with

$$\begin{aligned} \otimes &: (C_*(\tilde{N}, \mathbb{Z}) : \text{mod-}\mathbb{Z}) \\ &\rightarrow \left(\mathbb{C} \xrightarrow{\mathbb{Z}\text{-mod}} C_*(\tilde{N}, \mathbb{Z}) \right) \\ &\rightarrow (\mathbb{C} : \mathbb{Z}\text{-mod}) \\ &\xrightarrow{\mathbb{Z}\text{-mod}} \mathbb{Z}\text{-mod} \end{aligned}$$

$$v \otimes_{\kappa} \lambda^{-1}w = \kappa(\lambda)v \otimes w$$

Example:

$$\mathbb{R}^3 \otimes_{\rho} (\mathbb{R}^*)^3$$

$$\rho((a, b, c)) = \begin{pmatrix} b \\ a \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes_{\rho} (a, b, c) =$$

ℓ^2 -alexander torsion

Definition

- X connected, finite CW-complex
 - connected: because otherwise $\pi_1(N)$ is not well-defined (only for (X, \cdot))
 - finite: finitely many equivariant cells
 - equivariant G -cells: $G/H \times D^n$ is glued into the skeleton ($H \triangleleft G$)
 - $\pi = \pi_1(X)$
 - $\varphi \in H^1(X; \mathbb{R}) \cong \left\{ \begin{matrix} \pi \rightarrow \mathbb{R} \\ \text{Grp} \end{matrix} \right\}$
 - $\kappa : \left\{ \begin{matrix} \pi \rightarrow \mathbb{R} \\ \text{Grp} \end{matrix} \right\} \rightarrow \mathbb{R}_{>0} \rightarrow \left\{ \begin{matrix} \mathbb{Z}\pi \rightarrow \mathbb{R}\pi \\ \text{Ring} \end{matrix} \right\}$
- $$\kappa(\varphi, t)(g) = t^{\varphi(g)} g$$

$\kappa(\varphi, t)$ -twisted ℓ^2 -chain complex:

wie besser aufschreiben? $C_*^{(2)}(\tilde{X}; \kappa) : \text{Chain}_{\text{FinGen}}(\text{Hilbert-Mod})$

$\otimes_{\kappa(\varphi, t)}$? $C_*^{(2)}(\tilde{X}; \kappa) := \ell^2 \pi \otimes_{\kappa(\varphi, t)} C_*(\tilde{X})$

Definition

- only if $C_*^{(2)}(\tilde{X}; \kappa)$ is of *determinant class*
- pick *cellular basis* of \tilde{X}

ℓ^2 -torsion:

$$\rho^{(2)}(C_*^{(2)}) := \sum_{n \in \mathbb{Z}} (-1)^{n+1} \log \det_{\mathcal{R}(G)} d_n^{(2)}$$

Definition

- only if $C_*^{(2)}(\tilde{X}; \kappa)$ is ℓ^2 -acyclic (no reduced homology)

full ℓ^2 -Alexander torsion

was bedeutet das?

$$\begin{aligned} \tau^{(2)}(X, \varphi)(t) &:= \exp(-\rho^{(2)}(C_*^{(2)}(\tilde{X}; \kappa))) \\ &= \exp\left(-\sum_{n \in \mathbb{Z}} (-1)^{n+1} \log \det_{\mathcal{R}(G)} d_n^{(2)}\right) \\ &= \exp\left(\sum_{n \in \mathbb{Z}} (-1)^n \log \det_{\mathcal{R}(G)} d_n^{(2)}\right) \\ &= \prod_{n \in \mathbb{Z}} \exp((-1)^n \log \det_{\mathcal{R}(G)} d_n^{(2)}) \\ &= \prod_{n \in \mathbb{Z}} (\det_{\mathcal{R}(G)} d_n^{(2)})^{(-1)^n} \\ &= \dots \det_{\mathcal{R}(G)} d_{-4}^{(2)} \frac{1}{\det_{\mathcal{R}(G)} d_{-3}^{(2)}} \det_{\mathcal{R}(G)} d_{-2}^{(2)} \frac{1}{\det_{\mathcal{R}(G)} d_{-1}^{(2)}} \det_{\mathcal{R}(G)} d_0^{(2)} \frac{1}{\det_{\mathcal{R}(G)} d_1^{(2)}} \det_{\mathcal{R}(G)} d_2^{(2)} \dots \end{aligned}$$

Proposition

- picking another *cellular basis*
- \Rightarrow new *base change matrix* is *generalized permutation matrix* P with entries $\pm t^{\varphi(g_i)} g_i$

Proposition Fuglede-Kadison determinant of P (for $t \neq 1$) is

stimmt das?

base change matrix?

generalized permutation matrix?

$$t^{\varphi(g_1)+\dots+\varphi(g_{k_n})}$$

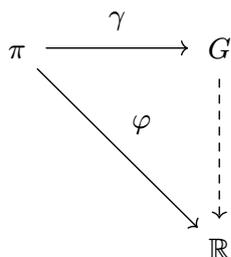
woher kommt hier k_n ?

Proposition

- N connected, compact irreducible 3-manifold
- $\#\pi_1(N) = \infty$
- ∂N empty or consists of incompressible tori

Definition

- "twist"
- go from \tilde{N} to ? using $\gamma : \pi \rightarrow_{\text{Grp}} G$ with



ℓ^2 -Alexander torsion

$$\tau^{(2)}(N, \gamma, \varphi) = \exp(-\rho^{(2)}(C_*^{(2)}(\tilde{X}; \kappa)))$$

$$\text{where } \kappa(\varphi, \gamma, t)(g) := t^{\varphi(g)}\gamma(g)$$

is $G \triangleleft \pi$?
currently only working with universal cover \tilde{N} , so maybe work with N_G ?

consists = $\bigcup \mathbb{T}_i$

incompressible tori?

are we working with different coverings now?

Bibliography

- [1] J. Dubois, S. Friedl, and W. Lück, “The L^2 -Alexander Torsion of 3-Manifolds.” Accessed: Jan. 14, 2026. [Online]. Available: <http://arxiv.org/abs/1410.6918>